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Counting vertices among all noncrossing trees by levels and degrees

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Abstract

In this paper, we enumerate vertices among all noncrossing trees based on their levels and degrees. Moreover, we obtain the number of eldest children, first children, non-first children, leaves and non-leaves which reside at a given level. We use generating functions, butterfly decomposition of noncrossing trees and bijections to arrive at our results.

Keywords: Noncrossing tree, level, degree, eldest child, first child, non-first child, leaf, non-leaf.

2020 MSC: 05C30, 05A19, 05C05.

1. Introduction

Enumerations of vertices according to levels and degrees have been investigated by various authors for classes of trees such are Cayley trees [9], plane trees [3, 8, 5, 14], t-ary trees [1, 10, 13] among other categories of trees. For plane trees and t-ary trees, the enumerations are based on the butterfly decomposition of plane trees introduced by Chen, Li and Shapiro in [2].

One of the properties of trees is that given any two vertices, there is always a unique path (sequence of edges) connecting the two vertices. The number of the edges on the path is the length of such a path. Given any rooted tree, the level of a vertex (sometimes called height of the vertex) is the length of the path from the root to the particular vertex. It is worth noting that the root of any tree resides on level 0. Consider a plane tree P. A vertex x of P is a child (respectively, a parent) of y if x is adjacent to y and x is at lower level (respectively, higher level) than y. Children of the same parent are siblings. On a given level, a vertex that is on the far left is the eldest child. A path from the root which connects the eldest children is a leftmost path. Given any vertex, a child which appear on far left is its first child and a vertex which is not a first child is said to be a non-first child.

Of interest, in this paper, is the set of noncrossing trees which was introduced by Noy [7] in 1998. These are trees drawn in the plane such that vertices are on the boundary of a circle and edges are line segments that do not cross inside the circle. In this paper, vertices of noncrossing trees are labelled in anticlockwise direction around the circle. In 2002, Panholzer and Prodinger [11] came up with the (l, r)-representation of noncrossing trees. Consider an edge (i, j) of a noncrossing tree such that i is the root or closes to the root in comparison with j. In the representation, if $j < i$ (respectively, $j > i$) then j is labelled l (respectively, r). The tree obtained is a plane tree with unlabelled root, all children of the root are labelled r, and the

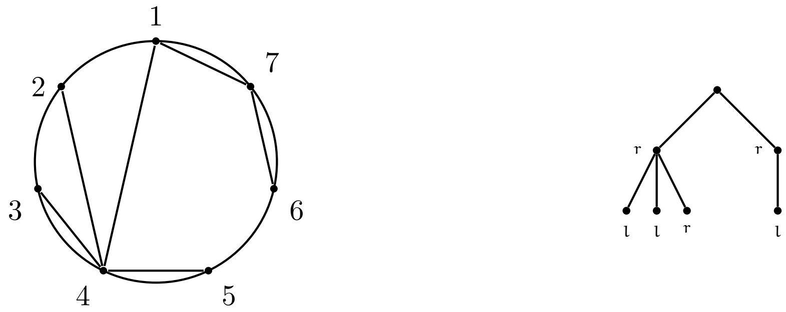

Figure 1: On the left is a noncrossing tree on 7 vertices and on the right is its $(\mathfrak{l}, \mathfrak{r})$-representation.

remaining vertices are labelled $\mathfrak{l}$ or $\mathfrak{r}$. In Figure 1, we get an example of a noncrossing tree together with its $(\mathfrak{l}, \mathfrak{r})$-representation.

In 2015, Okoth [9] enumerated noncrossing trees according to the level of a vertex such all the vertices on the path from the root (vertex 1) are all labelled $\mathfrak{r}$. The present study considers all paths where the vertices on the path are labelled $\mathfrak{r}$ or $\mathfrak{l}$. The degree of a vertex is the number of its children in the $(\mathfrak{l}, \mathfrak{r})$-decomposition of the noncrossing tree. A vertex of degree 0 is a leaf and a vertex which is not a leaf is a non-leaf.

In the last quarter century, butterfly decomposition introduced in 1999 by Flajolet and Noy [6] has been prominent in the enumeration of noncrossing trees. By a butterfly, we mean a pair of noncrossing trees which share vertex. Let $N(x)$ and $B(x)$ be respectively the generating function for noncrossing trees and butterflies where x marks non-root vertex. Then,

$$N(x) = \frac{1}{1 - B(x)} \quad \text{and} \quad B(x) = \frac{(xN(x))^2}{x}.$$

So,

$$N(x) = \frac{1}{1 - xN(x)^2}. \tag{1.1}$$

Setting $N(x) = \frac{M(x)}{\sqrt{x}}$, then equation (1.1) becomes

$$M(x) = \frac{\sqrt{x}}{1 - M(x)^2}. \tag{1.2}$$

Equation (1.2) is in a form we can apply Lagrange inversion formula [15] which we now state.

Theorem 1.1 (Lagrange inversion formula, [15]). Let $A(z)$ be a generating function that can be written as $A(z) = z\Gamma(A(z))$, where $A(0) \neq 0$. Then, $n[z^n]A(z)^m = m[\lambda^{n-m}]\Gamma(\lambda)^n$.

In this paper, we adopt the butterfly decomposition of plane trees in [2] to noncrossing trees. Consider a vertex j labelled $\mathfrak{r}$ (respectively, $\mathfrak{l}$) whose parent is vertex i in a noncrossing tree, then the children of i can be separated into three subsets i.e., the vertices labelled $\mathfrak{l}$, vertices labelled $\mathfrak{r}$ on the left of j and vertices labelled $\mathfrak{r}$ on the right of j (respectively, the vertices labelled $\mathfrak{l}$ on the left of j, vertices labelled $\mathfrak{l}$ on the right of j and vertices labelled $\mathfrak{r}$). This is illustrated by Figure 2.

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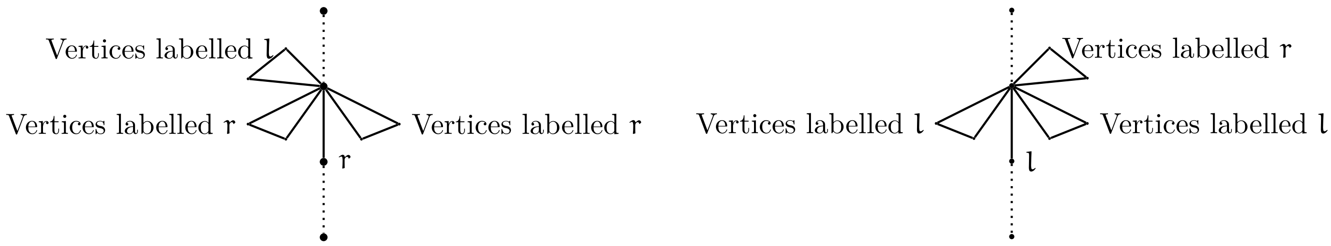

Figure 2: Decomposition of noncrossing trees based on the label of a child vertex.

The paper is organized as follows. We find explicit formulas and in some instances asymptotic results for the number of vertices, eldest children, first children and non-first children of given degrees that reside on a certain level in Section 2. We also enumerate leaves and non-leaves that are on a given label among all noncrossing trees. Bijections regarding these structures are established in Section 3 and the paper is concluded in Section 4 in which some problems are posed.

2. Enumerations

In this section, we enumerate the number of vertices in all noncrossing trees on a given label. We also use first children, eldest children, non-first children, leaves and non-leaves as the statistics of enumeration. We begin with the enumeration of vertices of all types.

2.1. Levels and degrees

In the sequel, we obtain a formula for the number of vertices of a given degree that reside on a certain level among all noncrossing trees with a given number of vertices.

Theorem 2.1. The number of vertices at level $k \geqslant 1$ with degree d in noncrossing trees on n vertices is given by

$$(d+1)2^{k-1} \cdot \frac{3k+2d-1}{2n+k-3}\binom{3n-d-5}{n-k-d-1}. \tag{2.1}$$

Proof. Noncrossing trees in which there is a path of length $k \geqslant 1$ such that the terminating vertex has degree d is decomposed as shown in Figure 3.

Based on the decomposition, the generating function for these trees is given by

$$(xN(x)^2)(xN(x)^3)^{k-1}x(xN(x)^2)^d = x^{k+d+1}N(x)^{3k+2d-1}.$$

We need to extract the coefficient of x^n in the generating function which we now perform.

$$[x^n]x^{k+d+1}N(x)^{3k+2d-1} = [x^{n-k-d-1}]N(x)^{3k+2d-1} = [x^{\frac{2n+k-3}{2}}]M(x)^{3k+2d-1}.$$

By Lagrange inversion formula (Theorem 1.1), we get

$$\begin{aligned}
[x^n]x^{k+d+1}N(x)^{3k+2d-1} &= \frac{3k+2d-1}{2n+k-3}[m^{2(n-k-d-1)}](1-m^2)^{-(2n+k-3)} \\
&= \frac{3k+2d-1}{2n+k-3}[m^{2(n-k-d-1)}]\sum_{i\geqslant 0}\binom{-(2n+k-3)}{i}(-m^2)^i \\
&= \frac{3k+2d-1}{2n+k-3}\binom{3n-d-5}{n-k-d-1}.
\end{aligned}$$

The child of the root is labelled r. Each vertex from level 2 till the last vertex can either be labelled r or l. So, there are 2^{k-1} choices for labels of vertices on the path. Moreover, there are $d+1$ choices for the labels of the children of the final vertex. This completes the proof. □

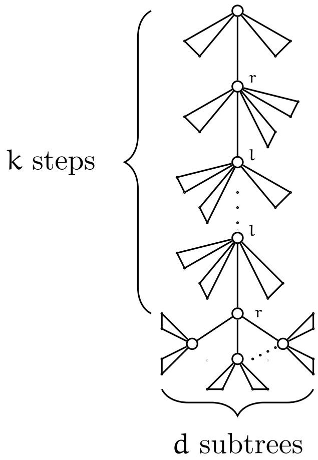

Figure 3: Decomposition of a noncrossing tree with a vertex of degree d at level $k \geqslant 1$.

Summing over all values of d in (2.1), we have the following corollary.

Corollary 2.2. There are

$$2^{k-1} \cdot \frac{3k+1}{2n+k-1}\binom{3n-3}{n-k-1} \tag{2.2}$$

vertices at level $k \geqslant 1$ in noncrossing trees on n vertices.

Setting $d = 0$ in (2.1), we find that there are

$$2^{k-1} \cdot \frac{3k-1}{2n+k-3}\binom{3n-5}{n-k-1} \tag{2.3}$$

leaves at level $k \geqslant 1$ in noncrossing trees with n vertices. On average, there are

$$\frac{(3k-1)(2n+k-1)(2n+k-2)}{(3k+1)(3n-3)(3n-4)} \tag{2.4}$$

vertices at level $k \geqslant 1$ in these trees on n vertices. This follows by dividing (2.3) by (2.2). This number tends to $\frac{4(3k-1)}{9(3k+1)}$ as n goes to infinity. Further setting $k = 1$ in (2.3) implies that the total number of leaves which are children of the root is

$$\frac{1}{n-1}\binom{3n-5}{n-2},$$

where n is the number of vertices in the noncrossing trees. Setting $k = 1$ in (2.4), we get that a noncrossing tree on n vertices has on average,

$$\frac{n(2n-1)}{(3n-3)(3n-4)}$$

leaf children, which tends to $\frac{2}{9}$ as n grows to infinity. The following result was proved by Okoth in his PhD thesis [9].

Corollary 2.3 ([9, Proposition 8.1.1]). The number of vertices at level $k \geqslant 1$ in noncrossing trees on n vertices such that all the vertices on the path from the root to the vertices in question are all labelled r is given by

$$\frac{3k+1}{2n+k-1}\binom{3n-3}{n-k-1}. \tag{2.5}$$

Proof. The result follows by mimicking the proof of Theorem 2.1 and not selecting the choices for labels of vertices on the path from the root to the vertex at level $k \geqslant 1$ since all these vertices are labelled r. We then sum over all d to get the desired result. □

Corollary 2.4 ([9, Corollary 8.1.3]). There are

$$\frac{2}{n}\binom{3n-3}{n-2}$$

children of the root in noncrossing trees on n vertices.

Proof. Set $k = 1$ in (2.5). □

2.2. Eldest children

In this subsection, we count eldest children of a given degree that are on a given level in the (l, r)-decomposition of noncrossing trees.

Theorem 2.5. The number of eldest children at level $k \geqslant 1$ with degree d in noncrossing trees on n vertices such that for each eldest child, there are ℓ vertices labelled l on the leftmost path from the root to the eldest child is given by

$$(d+1)\cdot\frac{k+\ell+2d}{2n-k+\ell-2}\binom{3n+\ell-2k-d-4}{n-k-d-1}\binom{k-1}{\ell}. \tag{2.6}$$

Proof. The decomposition of the trees is given in Figure 4.

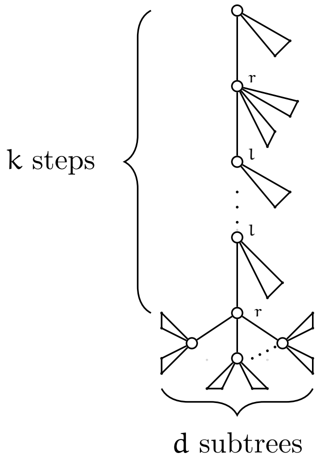

Figure 4: Decomposition of a noncrossing tree with an eldest child of degree d at level $k \geqslant 1$.

The generating function for these trees is thus

$$(xN(x))^{k-\ell}(xN(x)^2)^{\ell}x(xN(x)^2)^d = x^{k+d+1}N(x)^{k+\ell+2d}.$$

Making use of Lagrange inversion formula, we extract the coefficient of x^n in the generating function as follows:

$$[x^n]x^{k+d+1}N(x)^{k+\ell+2d} = [x^{n-k-d-1}]N(x)^{k+\ell+2d} = [x^{\frac{2n-k+\ell-2}{2}}]M(x)^{k+\ell+2d}$$
$$= \frac{k+\ell+2d}{2n-k+\ell-2}\binom{3n-2k+\ell-d-4}{n-k-d-1}.$$

There are $\binom{k-1}{\ell}$ choices for vertices on the leftmost path to be labelled l, i.e., all the vertices except the root and its first child cannot be labelled l. Also, there are $d+1$ choices for the labels of the children of the eldest child at level $k \geqslant 1$. Thus the proof. □

Summing over all values of d in (2.6), we obtain:

Corollary 2.6. The total number of eldest children that reside on level $k \geqslant 1$ in noncrossing trees on n vertices such that for each eldest child, there are ℓ vertices labelled l on the leftmost path from the root to the eldest child is

$$\frac{k+\ell+2}{2n-k+\ell}\binom{3n+\ell-2k-2}{n-k-1}\binom{k-1}{\ell}. \tag{2.7}$$

We obtain the following result upon setting $d = 0$ in (2.6).

Corollary 2.7. There are

$$\frac{k+\ell}{2n-k+\ell-2}\binom{3n+\ell-2k-4}{n-k-1}\binom{k-1}{\ell}$$

eldest children, which are also leaves, that reside on level $k \geqslant 1$ in noncrossing trees on n vertices such that for each eldest child, there are ℓ vertices labelled l on the leftmost path from the root to the eldest child.

Corollary 2.8. The number of eldest children at level $k \geqslant 1$ with degree d in noncrossing trees on n vertices such that for each eldest child, all the vertices on the leftmost path from the root to the eldest child are labelled r is given by

$$(d+1)\cdot\frac{k+2d}{2n-k-2}\binom{3n-2k-d-4}{n-k-d-1}. \tag{2.8}$$

Proof. We set $\ell = 0$ in (2.6). □

Further, summing over all values of d in (2.8) or setting $\ell = 0$ in (2.7) we get the following corollary.

Corollary 2.9. The total number of eldest children that reside on level $k \geqslant 1$ in noncrossing trees on n vertices such that for each eldest child, all the vertices on the leftmost path from the root to the eldest child are labelled r is

$$\frac{k+2}{2n-k}\binom{3n-2k-2}{n-k-1}. \tag{2.9}$$

By setting $d = 0$ in (2.8), we arrive at the following corollary.

Corollary 2.10. There are

$$\frac{k}{2n-k-2}\binom{3n-2k-4}{n-k-1} \tag{2.10}$$

eldest children, which are also leaves, that reside on level $k \geqslant 1$ in noncrossing trees on n vertices such that for each eldest child, all the vertices on the leftmost path from the root to the eldest child are labelled r.

On setting $k = 1$ in (2.10), we obtain the total number of eldest children, which are also leaves, of the root as

$$\frac{1}{2n-3}\binom{3n-6}{n-2},$$

which also counts noncrossing trees on $n-1$ vertices.

2.3. First children

In this subsection, we are interested in the enumeration of first children among all noncrossing trees with a specified number of vertices.

Theorem 2.11. The number of first children labelled l at level $k \geqslant 2$ with degree d in noncrossing trees on n vertices is given by

$$(d+1)2^{k-2} \cdot \frac{3k+2d-2}{2n+k-4}\binom{3n-d-6}{n-k-d-1}. \tag{2.11}$$

Proof. Noncrossing trees in which there is a path of length $k \geqslant 2$ such that the terminating vertex is a first child labelled l and has degree d is decomposed as shown in Figure 5.

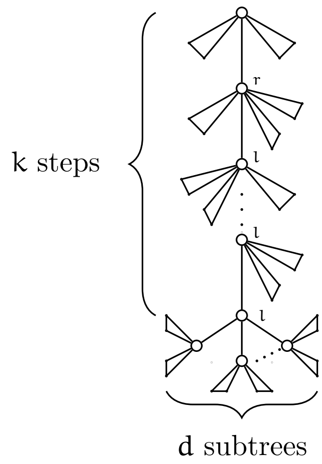

Figure 5: Decomposition of a noncrossing tree with a first child labelled l and of degree d at level $k \geqslant 2$.

The generating function for noncrossing trees in which the first child at level $k \geqslant 2$ is labelled l is thus given by

$$(xN(x)^2)(xN(x)^3)^{k-2}(xN(x)^2)x(xN(x)^2)^d = x^{k+d+1}N(x)^{3k+2d-2}.$$

We now use Lagrange inversion formula to extract the coefficient of x^n in the generating function:

$$[x^n]x^{k+d+1}N(x)^{3k+2d-2} = [x^{n-k-d-1}]N(x)^{3k+2d-2} = [x^{\frac{2n+k-4}{2}}]M(x)^{3k+2d-2}$$
$$= \frac{3k+2d-2}{2n+k-4}\binom{3n-d-6}{n-k-d-1}.$$

The child of the root is labelled r. Each vertex from level 2 until the second last vertex can either be labelled r or l. So, there are 2^{k-2} choices for labels of vertices on the path. In addition, there are $d+1$ choices for the labels of the children of the first child at level $k \geqslant 2$. So, the desired result is

$$(d+1)2^{k-2} \cdot \frac{3k+2d-2}{2n+k-4}\binom{3n-d-6}{n-k-d-1}.$$

□

Corollary 2.12. The total number of first children labelled l at level $k \geqslant 2$ in noncrossing trees on n vertices is

$$2^{k-2} \cdot \frac{k}{n-1}\binom{3n-3}{n-k-1}. \tag{2.12}$$

Proof. The result follows by summing over all values of d in (2.11). □

By setting $d = 0$ in (2.11), we obtain the following corollary:

Corollary 2.13. There are

$$2^{k-2} \cdot \frac{3k-2}{2n+k-4}\binom{3n-6}{n-k-1}$$

first children, which are also leaves, labelled l that reside on level $k \geqslant 2$ in noncrossing trees on n vertices.

Theorem 2.14. The number of first children labelled r at level $k \geqslant 2$ with degree d in noncrossing trees on n vertices is given by

$$(d+1)2^{k-2} \cdot \frac{3k+2d-3}{2n+k-5}\binom{3n-d-7}{n-k-d-1}. \tag{2.13}$$

Proof. Noncrossing trees in which there is a path of length $k \geqslant 2$ such that the terminating vertex is a first child labelled r and has degree d is decomposed as shown in Figure 6.

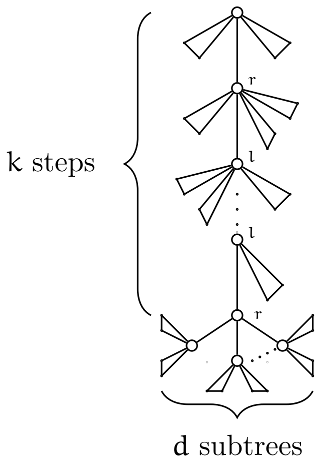

Figure 6: Decomposition of a noncrossing tree with a first child labelled r and of degree d at level $k \geqslant 2$.

So,

$$(xN(x)^2)(xN(x)^3)^{k-2}(xN(x))x(xN(x)^2)^d = x^{k+d+1}N(x)^{3k+2d-3}$$

is generating function for noncrossing trees in which the first child at level $k \geqslant 2$ is labelled r.

We have,

$$[x^n]x^{k+d+1}N(x)^{3k+2d-3} = [x^{n-k-d-1}]N(x)^{3k+2d-3} = [x^{\frac{2n+k-5}{2}}]M(x)^{3k+2d-3}$$
$$= \frac{3k+2d-3}{2n+k-5}\binom{3n-d-7}{n-k-d-1}.$$

The child of the root is labelled r. Each vertex from level 2 until the second last vertex can either be labelled r or l. So, there are 2^{k-2} choices for labels of vertices on the path. Also, there are $d+1$ choices for the labels of the children of the first child at level $k \geqslant 2$. The required formula is thus

$$(d+1)2^{k-2} \cdot \frac{3k+2d-3}{2n+k-5}\binom{3n-d-7}{n-k-d-1}.$$

□

Corollary 2.15. There are

$$2^{k-2} \cdot \frac{3k-3}{2n+k-5}\binom{3n-7}{n-k-1}$$

first children, which are also leaves, labelled r that reside on level $k \geqslant 2$ in noncrossing trees on n vertices.

Proof. Set $d = 0$ in (2.13). □

Summing over all values of d in (2.13), we get the following result.

Corollary 2.16. The total number of first children labelled r at level $k \geqslant 2$ in noncrossing trees on n vertices is

$$2^{k-2} \cdot \frac{3k-1}{3n-4}\binom{3n-4}{n-k-1}. \tag{2.14}$$

We obtain the total number of first children of degree d at level $k \geqslant 2$ in noncrossing trees with n vertices by adding (2.11) and (2.13). Moreover, the total number of first children at level $k \geqslant 2$ in noncrossing trees with n vertices is

$$2^{k-2} \cdot \frac{15nk - 2n - 19k + 3k^2 + 2}{(2n+k-2)(3n-4)}\binom{3n-4}{n-k-1}$$

which is arrived at by adding (2.12) and (2.14).

Proposition 2.17. There are

$$(d+1) \cdot \frac{2d+1}{2n-3}\binom{3n-d-6}{n-d-2} \tag{2.15}$$

first children, of degree d, of the root in noncrossing trees with n vertices.

Proof. The decomposition of these trees is as shown in Figure 7.

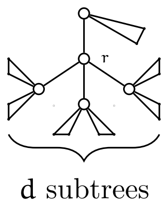

Figure 7: Decomposition of noncrossing trees with non-first children of degree d which are children of the root.

The generating function is thus $(xN(x))x(xN(x)^2)^d = x^{d+2}N(x)^{2d+1}$. So,

$$\begin{aligned}[x^n]x^{d+2}N(x)^{2d+1} &= [x^{n-d-2}]N(x)^{2d+1} = [x^{\frac{2n-3}{2}}]M(x)^{2d+1} \\ &= \frac{2d+1}{2n-3}\binom{3n-d-6}{n-d-2}.\end{aligned}$$

Since there are $d+1$ ways of distributing the labels l and r to the d children of the first child of the root, the proof follows. □

Summing over all values of d in (2.15), we obtain

$$\frac{1}{n-1}\binom{3n-3}{n-2}$$

as the total number of the first children of the root in noncrossing trees with n vertices, i.e., each noncrossing tree has exactly one first child of the root. On setting $d=0$ in (2.15), we obtain the total number of first children of the root which are also leaves in noncrossing trees on n vertices as

$$\frac{1}{2n-3}\binom{3n-6}{n-2}. \tag{2.16}$$

Formula (2.16) counts noncrossing trees with $n-1$ vertices. There is a simple combinatorial argument for this result: Consider a noncrossing tree on $n-1$ vertices. We obtain a noncrossing tree on n vertices, such that the first child of the root is leaf, from a noncrossing tree on $n-1$ vertices by attaching a new vertex to the root so that it becomes a first child of the root. The procedure is easily reversible.

2.4. Non-first children

In this subsection, our statistic of enumeration is non-first children. We consider two kinds of noncrossing trees i.e., those trees in which the labels of the non-first and its sibling on the immediate left are the same and when the labels are different.

Theorem 2.18. The number of non-first children at level $k \geqslant 2$ and of degree d in noncrossing trees on n vertices such that for each non-first child, the sibling on the immediate left of the non-first child has the same label as the non-first child is given by

$$(d+1)2^{k-1}\cdot\frac{3k+2d+1}{2n+k-3}\binom{3n-d-6}{n-k-d-2}. \tag{2.17}$$

Proof. The decompositions for the trees is given in Figure 8.

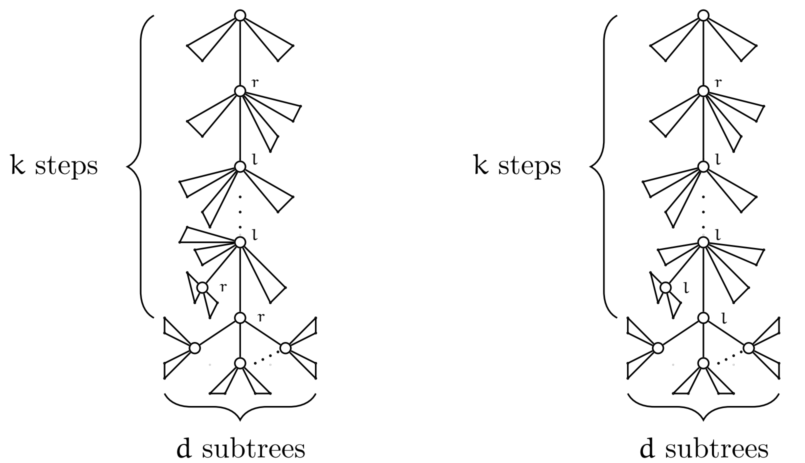

Figure 8: Decomposition of noncrossing trees with non-first children of degree d at level $k \geqslant 1$ such that the label of the non-first child and its immediate sibling on the left are the same.

From the decompositions, the generating function is

$$(xN(x)^2)(xN(x)^3)^{k-1}(xN(x)^2)x(xN(x)^2)^d = x^{k+d+2}N(x)^{3k+2d+1}.$$

We use Lagrange inversion formula to extract the coefficient of x^n.

$$[x^n]x^{k+d+2}N(x)^{3k+2d+1} = [x^{n-k-d-2}]N(x)^{3k+2d+1} = [x^{\frac{2n+k-3}{2}}]M(x)^{3k+2d+1}$$
$$= \frac{3k+2d+1}{2n+k-3}\binom{3n-d-6}{n-k-d-2}.$$

The child of the root is labelled r. Each vertex from level 2 until the terminating vertex can either be labelled r or l. So, there are 2^{k-1} choices for labels of vertices on the path. Also, there are $d+1$ choices for the labels of the children of the non-first child at level $k \geqslant 2$. The formula thus follows by product rule of counting. □

We set $d = 0$ in (2.17) to obtain:

Corollary 2.19. There are

$$2^{k-1} \cdot \frac{3k+1}{2n+k-3}\binom{3n-6}{n-k-2}$$

non-first children, which are also leaves, at level $k \geqslant 2$ in noncrossing trees on n vertices such that for each non-first child (leaf), the sibling on the immediate left of the non-first child has the same label as the non-first child.

By summing over all values of d in (2.17) , we get the following result.

Corollary 2.20. The total number of non-first children at level $k \geqslant 2$ in noncrossing trees on n vertices such that for each non-first child, the sibling on the immediate left of the non-first child has the same label as the non-first child is

$$2^{k-1} \cdot \frac{3k+3}{2n+k-1}\binom{3n-3}{n-k-2}. \tag{2.18}$$

Theorem 2.21. The number of non-first children at level $k \geqslant 2$ and of degree d in noncrossing trees on n vertices such that for each non-first child labelled r, the sibling on the immediate left of the non-first child is labelled l is given by

$$(d+1)2^{k-2} \cdot \frac{3k+2d}{2n+k-4}\binom{3n-d-7}{n-k-d-2}. \tag{2.19}$$

Proof. The decomposition for the trees in question is as shown in Figure 9.

Based on the decomposition, the generating function is

$$(xN(x)^2)(xN(x)^3)^{k-2}(xN(x)^2)(xN(x)^2)x(xN(x)^2)^d = x^{k+d+2}N(x)^{3k+2d}.$$

Using Lagrange inversion formula, we obtain the coefficient of x^n :

$$[x^n]x^{k+d+2}N(x)^{3k+2d} = [x^{n-k-d-2}]N(x)^{3k+2d} = [x^{\frac{2n+k-4}{2}}]M(x)^{3k+2d}$$
$$= \frac{3k+2d}{2n+k-4}\binom{3n-d-7}{n-k-d-2}.$$

There are 2^{k-2} choices for labels of vertices on the path since the final vertex is labelled r. Also, there are $d+1$ choices for the labels of the children of the non-first child at level $k \geqslant 2$. The result thus follows. □

Corollary 2.22. There are

$$2^{k-2} \cdot \frac{3k+1}{2n+k-3}\binom{3n-6}{n-k-2}$$

non-first children, which are also leaves, at level $k \geqslant 2$ in noncrossing trees on n vertices such that for each non-first child (leaf) labelled r, the sibling on the immediate left of the non-first child is labelled l.

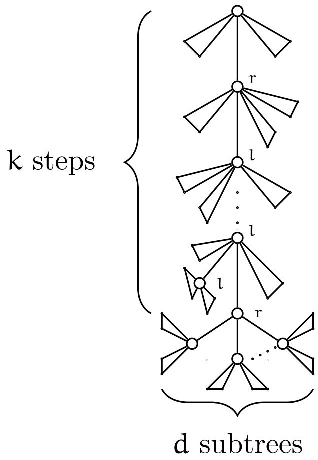

Figure 9: Decomposition of noncrossing trees with non-first children of degree d at level $k \geqslant 1$ such that the label of the non-first child and its immediate sibling on the left are different.

Proof. Set $d = 0$ in (2.19). □

Summing over all values of d in (2.19) , we get the following corollary.

Corollary 2.23. The total number of non-first children at level $k \geqslant 2$ in noncrossing trees on n vertices such that for each non-first child labelled r, the sibling on the immediate left of the non-first child is labelled l is

$$2^{k-2} \cdot \frac{3k+2}{2n+k-2}\binom{3n-4}{n-k-2}. \tag{2.20}$$

The total number of non-first children of degree d at level $k \geqslant 2$ in noncrossing trees on n vertices is obtained by adding (2.17) and (2.19). Moreover, the number of non-first children at level $k \geqslant 2$ in noncrossing trees with n vertices is the sum of (2.18) and (2.20).

Proposition 2.24. There are

$$(d+1) \cdot \frac{d+2}{n-1}\binom{3n-d-6}{n-d-3} \tag{2.21}$$

non-first children, of degree d, of the root in noncrossing trees with n vertices.

Proof. The decomposition of these trees is depicted in Figure 10.

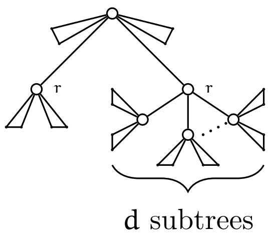

Figure 10: Decomposition of noncrossing trees with non-first children of degree d which are children of the root.

So, the generating function is $(xN(x)^2)(xN(x)^2)x(xN(x)^2)^d = x^{d+3}N(x)^{2d+4}$.

We have,

$$[x^n]x^{d+3}N(x)^{2d+4} = [x^{n-d-3}]N(x)^{2d+4} = [x^{\frac{2n-2}{2}}]M(x)^{2d+4}$$
$$= \frac{2d+4}{2n-2}\binom{3n-d-6}{n-d-3}.$$

Since there are $d+1$ ways of distributing the labels l and r to the d children of the non-first child of the root, the proof follows. □

Summing over all values of d in (2.21), we obtain

$$\frac{3}{n}\binom{3n-4}{n-3} \tag{2.22}$$

as the total number of the non-first children of the root in noncrossing trees with n vertices. Dividing (2.22) by

$$\frac{1}{n-1}\binom{3n-3}{n-2},$$

we find that on average there are $\frac{n-2}{n}$ non-first children of the root in a noncrossing tree with n vertices.

On setting $d = 0$ in (2.21), we obtain the total number of non-first children of the root which are also leaves in noncrossing trees on n vertices as

$$\frac{2}{n-1}\binom{3n-6}{n-3}.$$

The number of noncrossing trees on n vertices such that the root has degree $d+2$ was obtained by Deutsch and Noy [4] as

$$\frac{d+2}{n-1}\binom{3n-d-6}{n-d-3}.$$

Formula (2.21) thus gives the number of noncrossing trees on n vertices with root degree $d+2$ such that exactly one non-first child of the root is marked. We prove this fact by means of a bijection.

Proposition 2.25. There is a bijection between the set of noncrossing trees on n vertices with root degree $d+2$ such that exactly one non-first child of the root is marked and the set of noncrossing trees on n vertices such that there is a non-first child, of degree d, of the root.

Proof. Consider a noncrossing tree on n vertices with root degree $d+2$ such that there is a non-first child which is marked with an asterisk. We obtain the corresponding noncrossing tree on n vertices such that there is a non-first child, of degree d, of the root by the following procedure:

(i) Relabel the children of the marked vertex on the left wing as r.

(ii) Delete the left (respectively, right) wing of the butterfly rooted at the marked vertex and attach it at the root such that the the wing is on the left (respectively, right) of all the initial children of the root.

(iii) Detach all the subtrees rooted at the children of the root on the left (respectively, right) of marked vertex and attach them to the marked vertex as a left (respectively, right) wing of the butterfly rooted at the marked vertex.

The tree obtained is a noncrossing tree on n vertices such that the marked vertex corresponds to a non-first child of degree d.

We now reverse the procedure: Consider a noncrossing tree on n vertices such that there is non-first child i of the root with degree d. We obtain a corresponding a noncrossing tree on n vertices with root degree $d+2$ such that there is a non-first child of the root which is marked with an asterisk as follows.

(i) Mark i with an asterisk. Let the sibling of i on the immediate left as j.

(ii) Delete the left wing of the butterfly rooted at the marked vertex and attach it to root on the immediate left wing of the marked vertex. Relabel the children of the root as r. Moreover, delete the subtrees rooted at the children of the root on the left of j and attach them as a left wing of the butterfly rooted at i.

(iii) Delete the right wing of the butterfly rooted at the marked vertex and attach it to root on the immediate right wing of the marked vertex. Then, delete the subtrees rooted at the children of the root that were initially on the right of i and attach them as a right wing of the butterfly rooted at i.

This procedure is illustrated in Figure 11.

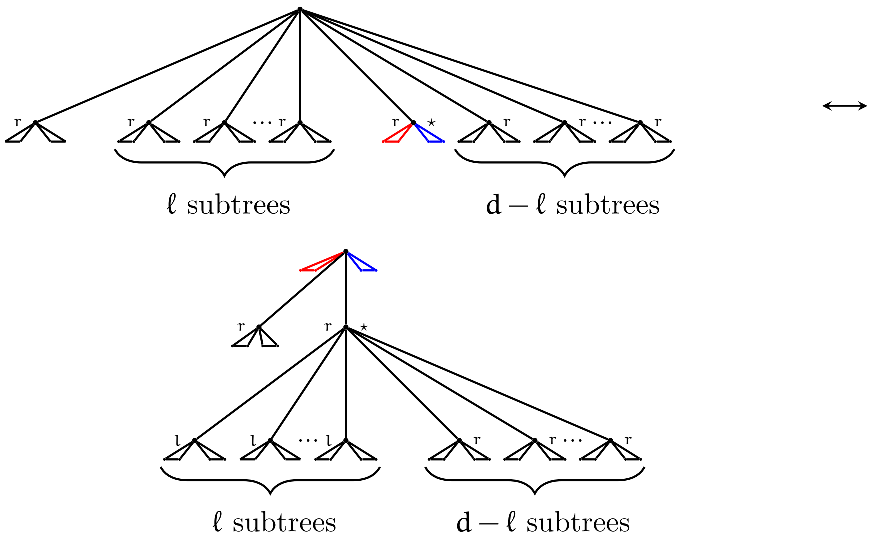

Figure 11: A noncrossing tree with root degree $d+2$ such that one of the non-first children of the root is marked and its corresponding noncrossing tree with a non-first child of the root having degree d.

□

Further bijections are given in Section 3.

2.5. Non-leaves

In this subsection, we obtain the total number of non-first children in all noncrossing trees.

Theorem 2.26. The number of non-leaf vertices in noncrossing trees on n vertices which reside on level $k \geqslant 1$ is given by

$$2^k \cdot \frac{3k+4}{2n+k}\binom{3n-3}{n-k-2}. \tag{2.23}$$

Proof. The trees are decomposed as given in Figure 12.

The generating function is thus

$$(xN(x)^2)(xN(x)^3)^k(xN(x)^2) = x^{k+2}N(x)^{3k+4}.$$

Now,

$$\begin{aligned}[x^n]x^{k+2}N(x)^{3k+4} &= [x^{n-k-2}]N(x)^{3k+4} = [x^{\frac{2n+k}{2}}]M(x)^{3k+4}\\ &= \frac{3k+4}{2n+k}\binom{3n-3}{n-k-2}.\end{aligned}$$

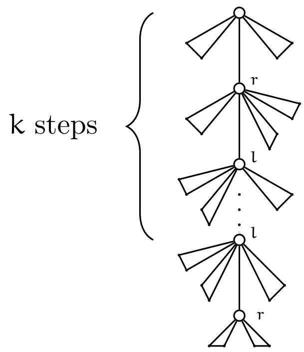

Figure 12: Decomposition of a noncrossing tree with a non-leaf vertex at level $k \geqslant 2$.

Since there are 2^k choices for the labels of the vertices on the path, the formula follows. □

We remark that (2.23) also gives the number of vertices that reside on level $k \geqslant 1$ in noncrossing trees on n vertices. By setting $k = 1$ in (2.23), we obtain:

Corollary 2.27. There are

$$\frac{14}{2n+1}\binom{3n-3}{n-3}$$

non-leaf vertices which are children of the root in noncrossing trees on n vertices.

3. Bijections

3.1. Eldest children and noncrossing trees of given root degree

Consider noncrossing trees with root degree d. The trees are decomposed as shown in Figure 13.

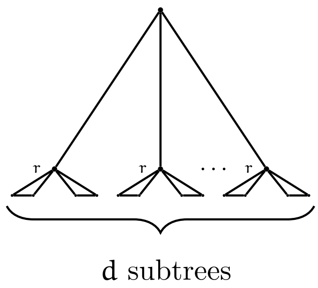

Figure 13: A noncrossing tree with root degree d.

The generating function is $x(xN(x)^2)^d = x^{d+1}N(x)^{2d}$. By Lagrange inversion formula, we have

$$\begin{aligned}[x^n]x^{d+1}N(x)^{2d} &= [x^{n-d-1}]N(x)^{2d} = [x^{\frac{2n-2}{2}}]M(x)^{2d} \\ &= \frac{d}{n-1}\binom{3n-d-4}{n-d-1}. \end{aligned} \tag{3.1}$$

This result was also obtained by Deutsch and Noy in [4]. Setting $d = k+1$ and $n = n-k+1$ in (3.1), we get

$$\frac{k+1}{n-k}\binom{3n-4k-2}{n-2k-1}$$

which is the same formula obtained by setting $k = 2k$ in (2.9). We therefore get the following result.

Proposition 3.1. There is a bijection between the set of noncrossing trees on $n-k+1$ vertices with root degree $k+1$ and the set of eldest children that reside on level $2k \geqslant 2$ in noncrossing trees on n vertices such that for each eldest child, all the vertices on the leftmost path from the root to the eldest child are labelled r.

Proof. Consider a noncrossing tree on $n-k+1$ vertices with root degree $k+1$. Let the children of the root be labelled $1, 2, \ldots, k+1$ from left to right. There are butterflies (with two wings) rooted at each child of the root. Now, consider a path of length $2k+1$ labelled $1, 2, \ldots, 2k+1$ such that vertex 1 will become the root of the tree being constructed. Attach each right wing (respectively, left wing) of the butterfly rooted at i as a right wing of the butterfly rooted at $2i-1$ (respectively, $2i$) in the tree under construction, where $i = 1, 2, \ldots, k$. Finally, attach the butterfly rooted at vertex $k+1$ in the noncrossing tree to vertex $2k+1$ of the path. Vertex $k+1$ in the noncrossing tree is the eldest child at level $2k \geqslant 2$ in the tree constructed. Each child of the root, labelled $1, 2 \ldots, k$, in the noncrossing tree contributes an extra vertex to the tree constructed. Moreover, the root of the noncrossing tree does not contribute to any vertex. Thus the total number of vertices in the tree constructed is $(n-k+1)+k-1 = n$.

For the reverse procedure, label the vertices on the path from the root to the eldest child at level $2k$ as $1, 2, \ldots, 2k+1$. We start with a vertex which is a root of the new tree under construction. For $i = 1, 2, \ldots, k$, attach a butterfly rooted at vertex $2i-1$ (respectively, $2i$) as left (respectively, right) wing of the butterfly rooted at the i^{th} child (counted from left to right) of the root of the tree under construction. This contributes k to the root degree. Finally, attach the butterfly rooted at eldest child at level $2k$ as the butterfly rooted at the first child of the root vertex. This contributes 1 to the root degree. The total root degree is thus $k+1$. Each disjoint pair of vertices in the path from the root to the eldest child at level $2k-1$ contributes a vertex to the tree constructed. The new vertex also contributes a vertex. So, the tree constructed has $n-k+1$ vertices with root degree $k+1$. The bijection is illustrated in Figure 14.

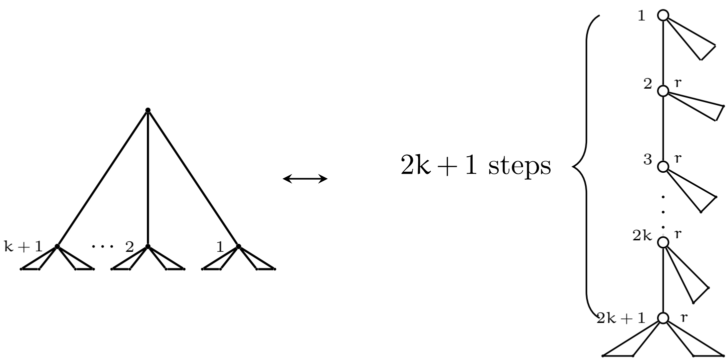

Figure 14: A noncrossing tree with degree $k+1$ and a corresponding eldest child at level $2k+1$.

□

3.2. First children and vertices on a certain level

Proposition 3.2. There is a bijection between the set of first children labelled l of degree d at level $k \geqslant 2$ in noncrossing trees on n vertices and the set of vertices of degree $d+1$ at level $k-1$ with first children of the vertices labelled l in noncrossing trees with n vertices.

Proof. Consider a noncrossing tree on n vertices such that there is a first child i labelled l of degree d at level $k \geqslant 2$. Let j be the parent of i and t be the parent of j. Now, delete the edge (t, j) and attach vertex j to i so that j is the first child of i. Relabel vertices j as l. Create an edge from t to i and relabel i so that it has the initial label of j. The tree obtained is a noncrossing tree on n vertices such that there is a vertex of degree $d+1$ at level $k-1$ and the first child of this vertex is labelled l.

The reverse procedure is as follows: Consider a noncrossing tree on n vertices such that there is a vertex of degree $d+1$ at level $k-1$ and the first child of this vertex is labelled l. Let the vertex be i and its first child be j. Also, let the parent of i be t. Delete edges (i, t) and (i, j) and create an edge from t to j. Give vertex j the initial label of i. Moreover, create an edge from j to i such that i is a first child of j. Vertex i is therefore at level k. Relabel vertex i as l. This procedure is described in Figure 15.

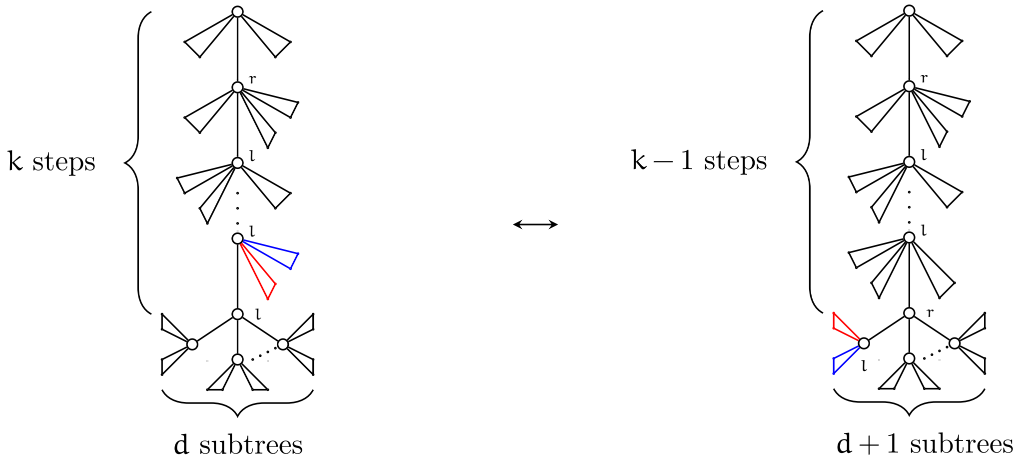

Figure 15: A noncrossing tree with a first child labelled l of degree d at level k and a corresponding a noncrossing tree with a vertex of degree $d + 1$ at level $k - 1$ such that the first child of the vertex is labelled l.

□

3.3. First children and non-first children

Based on Theorems 2.14 and 2.18 we prove the following result.

Proposition 3.3. There is a bijection between the set of first children labelled l of degree d at level $k \geqslant 2$ in noncrossing trees on n vertices and the set of non-first children of degree d at level $k - 1$ in noncrossing trees with n vertices such that for each non-first child, the sibling on the immediate left of the non-first child has the same label as the non-first child.

Proof. Consider a noncrossing tree on n vertices such that there is a first child i labelled l of degree d at level $k \geqslant 2$. Let j be the parent of i and t be the parent of j. Now, delete the edge (i, j) and attach vertex i to t so that j is on the left of i. Vertex i is therefore a non-first child of t. Relabel i so that it has the same label as j. The tree obtained is a noncrossing tree on n vertices such that there is a non-first child of degree d at level $k - 1$ and the sibling on the immediate left of the non-first child has the same label as the non-first child.

Conversely, consider a noncrossing tree on n vertices such that there is a non-first child of degree d at level $k - 1$ and the sibling on the immediate left of the non-first child has the same label as the non-first child. Let the non-first child be i, its parent be t and the sibling of i on its immediate left be j. Delete the edge (i, t) and create an edge from t to i such that i is a first child of j. Vertex i is therefore at level k. Relabel vertex i as l. Figure 16 is a depiction of the procedure.

□

Setting $d = 0$ in Proposition 3.3, we arrive at:

Corollary 3.4. There is a bijection between the set of first children labelled l, which are also leaves, at level $k \geqslant 2$ in noncrossing trees on n vertices and the set of non-first children, which are also leaves, at level $k - 1$ in noncrossing trees with n vertices such that for each non-first child, the sibling on the immediate left of the non-first child has the same label as the non-first child.

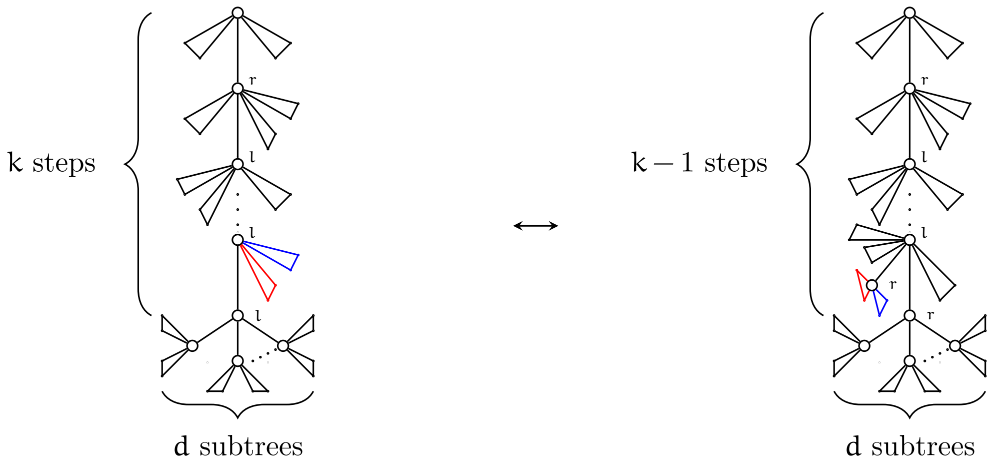

Figure 16: A noncrossing tree with a first child labelled l of degree d at level k and a corresponding a noncrossing tree with a non-first child of degree $d+1$ at level $k-1$ such that the label of the non-first child and its immediate sibling on the left are of the same label.

Proposition 3.5. There is a bijection between the set of first children labelled l of degree $d+1$ at level $k \geqslant 2$ in noncrossing trees on n vertices such that the first child of the first child is also labelled l and the set of non-first children of degree d at level k in noncrossing trees with n vertices such that for each non-first child, the sibling on the immediate left of the non-first child has a different label as the non-first child.

Proof. Consider a noncrossing tree on n vertices such that there is a first child i labelled l of degree $d+1$ at level $k \geqslant 2$ such that the first child of i is also labelled l. Let j be the first child of i and t be the parent of i. Now, delete the edge (i, j) and attach vertex j to t so that j is on the left of i. Vertex i is therefore a non-first child of t. Relabel the non-first child i from l to r. Delete all the subtrees rooted at the children of t, labelled l, which appear on the right of i and attach then to t, in order from left to right, so that they are on the left of j. The tree obtained is a noncrossing tree on n vertices such that there is a non-first child of degree d at level k and the sibling on the immediate left of the non-first child has a different label (l) as the non-first child r.

For inverse procedure, consider a noncrossing tree on n vertices such that there is a non-first child i of degree d at level k which is labelled r and the sibling on the immediate left of the non-first child is labelled l. Let the parent of i be t and the sibling of i on its immediate left be j. Delete the edge (j, t) and create an edge from j to i such that j is a first child of i. Moreover, delete the edge (i, t) and create an edge from i to t such that i is a first child of t. Vertex i retain its level k. Relabel vertex i as l. Figure 17 is an illustration of this bijection.

□

3.4. Vertices and non-leaves

The number of vertices on level k in noncrossing trees on n vertices is given by (2.2), i.e.,

$$2^{k-1} \cdot \frac{3k+1}{2n+k-1}\binom{3n-3}{n-k-1}. \tag{3.2}$$

Setting $k = k+1$ in (3.2), we find

$$2^{k} \cdot \frac{3k+4}{2n+k}\binom{3n-3}{n-k-2}$$

which also counts non-leaves on level k in noncrossing trees on n vertices as given by (2.23). The non-leaf vertex at level k is adjacent to a vertex on level $k+1$ otherwise it is a leaf. So each non-leaf vertex on level

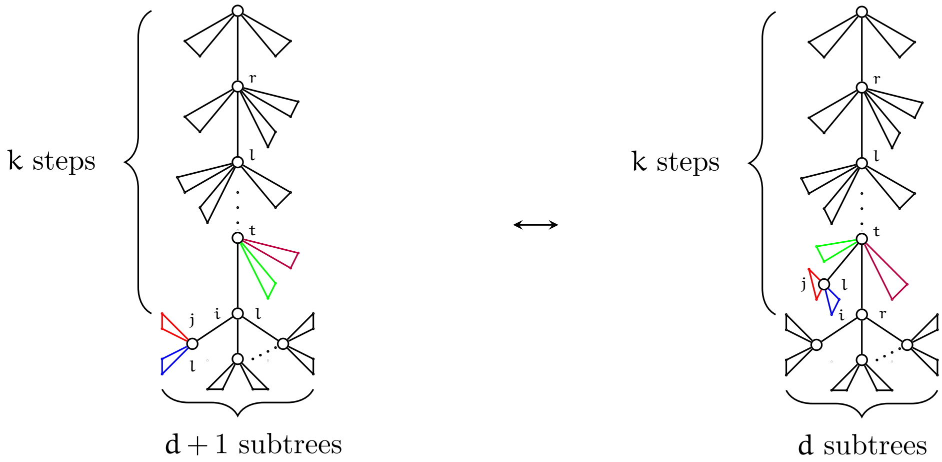

Figure 17: A noncrossing tree with a first child labelled l of degree $d+1$ at level k such that the first child of this first child is also labelled l and a corresponding a noncrossing tree with a non-first child of degree d at level k such that the label of the non-first child is r and its immediate sibling on the left is labelled l.

k has a unique vertex at level $k+1$. Also, any vertex on level $k+1$ must be connected to a vertex on level k which must be a non-leaf vertex by virtue of being connected to a vertex on level $k+1$. This proves the following corollary.

Proposition 3.6. There is a bijection between the set of vertices that reside on level $k+1$ in noncrossing trees on n vertices and the set of non-leaves that reside on level k in noncrossing trees on n vertices.

4. Conclusion

In this paper, we have enumerate vertices of noncrossing trees that reside on a given level if the vertices are eldest children, first children, non-first children, leaves and non-leaves. We arrived at our results by decomposition of trees. Moreover, we also created various bijections regarding these trees. Noncrossing trees have been generalized by giving labels $1, 2, \ldots, k$ to the vertices of noncrossing trees such that if (x, y) is an edge in the (l, r)-decomposition of the noncrossing trees of types (l, r) and (r, r) where x is closest to the root, x and y are labelled i and j respectively then $i + j \leqslant k + 1$. These trees are called k-noncrosssing trees introduced by Pang and Lv [12] in 2010. The case $k = 2$, was introduced earlier by Yan and Liu in [16]. It would be interesting to determine the number of vertices, first children, non-first children, eldest children, leaves and non-leaves in these trees that are on a given level.

References

[1] S. A. Abayo, I. O. Okoth, D. M. Kasyoki, Reachability in complete t-ary trees, Annals of Mathematics and Computer Science 18 (2023), 67–89. 1
[2] W. Y. C. Chen, N. Y. Li, L. W. Shapiro, The butterfly decomposition of plane trees, Discret. Appl. Math., 155(17) (2007), 2187–2201. 1, 1
[3] N. Dershowitz, S. Zaks, Enumerations of ordered trees, Discrete Math., 31(1) (1980), 9–28. 1
[4] E. Deutsch, M. Noy, Statistics on non-crossing trees, Discrete Math., 254(1-3) (2002), 75 – 87. 2.4, 3.1
[5] S-P. Eu, S. Seo, H. Shin, Enumerations of vertices among all rooted ordered trees with levels and degrees, Discrete Math., 340(9) (2017), 2123–2129. 1
[6] P. Flajolet, M. Noy, Analytic combinatorics of non-crossing configurations, Discrete Math., 204(1-3) (1999), 203 –229. 1
[7] M. Noy, Enumeration of noncrossing trees on a circle, Discrete Math., 180(1-3) (1998), 301-313. 1
[8] A. O. Nyariaro, I. O. Okoth, Reachability Results in Plane Trees, Commun. Adv. Math. Sci., 4(2) (2021), 75–88. 1
[9] I. O. Okoth, Combinatorics of oriented trees and tree-like structures, PhD Thesis, Stellenbosch University, 2015. 1, 1, 2.1, 2.3, 2.4
[10] I. O. Okoth, A. O. Nyariaro, Reachability results in labelled t-ary trees, Open J. Math. Sc, 5(1) (2021), 360-370. 1

[11] A. Panholzer, H. Prodinger, Bijection for ternary trees and non-crossing trees, Discret. Math., 250(1-3) (2002), 115–125. 1
[12] S. X. M. Pang, L. Lv, K-Noncrossing Trees and K-Proper Trees, 2010 2nd International Conference on Information Engineering and Computer Science, Wuhan, (2010), 1–3. 4
[13] S. Seo, H. Shin, A refined enumeration of p-ary labelled trees, Korean J. Math., 21(4) (2013), 495-502. 1
[14] S. Seunghyun, H. Shin, A refinement for ordered labeled trees, Korean J. Math., 20(2) (2012), 225-261. 1
[15] R. P. Stanley, Enumerative Combinatorics, Vol. 2, volume 62 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1999. 1, 1.1
[16] S. H. F. Yan, X. Liu, 2-noncrossing trees and 5-ary trees, Discrete Math., 309 no. 20 (2009) 6135–6138. 4